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Extension of Covariant POV-Measures in von Neumann Algebras

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We discuss the extension of POV-measures transforming covariantly with respect to automorphic group representations on von Neumann algebras.

1. INTRODUCTION

Systems of imprimitivity in the sense of Mackey (1949) consist of a projection-valued measure E on some G-space X and a unitary representation U of G fulfilling the covariance condition

$$U_g E(\Delta) U_g^* = E(g\Delta), \qquad \Delta \in \Sigma(X), \quad g \in G$$

More generally, one introduces systems of covariance by replacing the projection-valued (PV) measure with a positive-operator-valued (POV) measure. An imprimitivity theorem for systems of covariance can also be shown (Neumann, 1972; Scutaru, 1977; Cattaneo, 1979; Castrigiano and Henrichs, 1980; Holevo, 1982; Ali, 1984): one extends, in the spirit of Naimark (1943), the POV-measure to a PV-measure on a larger Hilbert space, and then constructs a group representation on the enlarged Hilbert space in order to arrive at a system of imprimitivity. The original system of covariance can then be regarded as resulting from a projection of the system of imprimitivity to the original smaller Hilbert space. In particular, if we have a system of covariance on a transitive G-space, then the original group representation is a subrepresentation of an induced representation.

In Breuer (1994a) systems of covariance were introduced in a more general context. The positive operators in the range of the POV-measure are not taken from $\mathcal{B}(\mathcal{H})$, but from an arbitrary von Neumann algebra. Further-

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more, one replaces the unitary ray representation with an automorphic group representation on the von Neumann algebra. In Section 2 we discuss the relevance of this generalization. In Section 3 we present an extension and an imprimitivity theorem for these more general systems of covariance.

2. GENERALIZED SYSTEMS OF COVARIANCE

Definition 1. A *W^*-system* $(\mathcal{M}, G, \alpha)$ consists of a W^*-algebra $\mathcal{M}$ with separable predual, a locally compact, separable group G, and a representation $\alpha: G \to \mathrm{Aut}(\mathcal{M})$ of G as a group of automorphisms of $\mathcal{M}$ such that:

(i) $\alpha_{g_1 g_2} = \alpha_{g_1}\alpha_{g_2}$.
(ii) For all operators $x \in \mathcal{M}$ the function $g \mapsto \alpha_g(x)$ is σ-weakly continuous.

The action α on $\mathcal{M}$ is said to be *ergodic* if $\alpha_g(x) = x$ for all $g \in G$ implies that x is a multiple of the identity operator.

The assumption that $\mathcal{M}$ has separable predual is equivalent (Pedersen, 1979, 3.9.9) to the fact that $\mathcal{M}$ is isomorphic to a von Neumann algebra on a separable Hilbert space. Although for some of the mathematical results this assumption is not necessary, it is usually made in physical applications. Similarly, the assumption that G is separable is not necessary for most of the results (Blattner, 1961, 1962). It can be dropped if one is willing to enter into more intricate mathematical arguments.

Example 1. The systems of traditional quantum mechanics can be regarded as W^*-systems in the following way. A quantum mechanical system is specified by a unitary ray representation U of a kinematical group G, which is, for example, the Poincaré group or the Galilei group. Associated to U is a representation α of G as a group of automorphisms of $\mathcal{B}(\mathcal{H})$ defined by

$$\alpha_g(x) := U_g x U_g^*, \qquad g \in G, \quad x \in \mathcal{B}(\mathcal{H})$$

Then $(\mathcal{B}(\mathcal{H}), G, \alpha)$ is a W^*-system. The choice of $\mathcal{B}(\mathcal{H})$ for $\mathcal{M}$ is a consequence of von Neumann's irreducibility postulate.

Conversely, every W^*-system, where $\mathcal{M}$ is a type I factor, can be brought into the form $(\mathcal{B}(\mathcal{H}), G, U \cdot U^*)$. This is due to the fact that all automorphisms α of a type I factor are inner: they are induced by a unitary operator $U \in \mathcal{M}$ by $\alpha(x) = UxU^*$.

Example 2. Let X be a G-space and μ a quasi-invariant measure on X. Denote by $L^\infty(X, \mu)$ the von Neumann algebra of μ-essentially bounded functions on X. G acts on $L^\infty(X, \mu)$ from the left by

$$\mathrm{Ad}\lambda(g)f(x) := f(g^{-1}x), \qquad f \in L^\infty(X, \mu)$$

$(L^\infty(X, \mu), G, \mathrm{Ad}\lambda)$ is a commutative W^*-system.

As these examples show, W^*-systems generalize quantum mechanical and classical systems. They can also describe quantum systems with superselection rules.

Generalized Systems of Covariance. Using W^*-systems, one arrives at a generalized notion of systems of covariance, which will be used in the sequel.

Definition 2. Let $(\mathcal{M}, G, \alpha)$ be a W^*-system and let X be a standard Borel G-space. A quasi-invariant POV-measure a on X with values in $\mathcal{M}$, together with the automorphic representation α of G on $\mathcal{M}$, is called a *system of covariance* (a, α) *based on* X if a acts covariantly with respect to α,

$$\alpha_g(a(\Delta)) = a(g\Delta), \qquad \Delta \in \Sigma(X), \quad g \in G$$

If G acts transitively on X, then (a, α) is called a *transitive* generalized system of covariance.

Traditionally systems of covariance are defined to consist of a unitary ray representation of G and a covariant POV-measure on X. Why is Definition 2 more general? The main reason is that an automorphic representation of G on $\mathcal{M}$ cannot always be replaced by a unitary ray representation of G on $\mathcal{H}$. I will briefly make some remarks on which automorphic group representations can be replaced with a unitary ray representation and which cannot.

Unitary Implementability of Automorphic Group Representations. An automorphic group representation α cannot be replaced with a unitary ray representation if there is some α_g which is not spatial. [An automorphism α of a von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is called spatial if there is a unitary operator U on $\mathcal{H}$, U not necessarily in $\mathcal{M}$, such that $\alpha = U \cdot U^*$.] Let me quote some partial results on when automorphisms are spatial. If $\mathcal{M}$ is a factor of type I or III, any automorphism is spatial. But if $\mathcal{M}$ is a factor of type II_∞ with commutant of type II_1, then there are examples of automorphisms which are not spatial, even if $\mathcal{H}$ is separable (Kadison and Ringrose, 1986, 13.4.3). It is possible to give a necessary and sufficient condition (Kadison and Ringrose, 1986, 9.6.33) when an automorphism of a type II_∞ factor with commutant of type II_1 is spatial. If $\mathcal{M}$ is not a factor, then one can guarantee that an automorphism is spatial if the central decomposition of $\mathcal{M}$ does not contain any factor of type II and if the automorphism leaves every central element fixed. This last condition is a severe restriction in physical applications. It can be dropped if, for example, the commutant of $\mathcal{M}$ is Abelian (Dixmier, 1959, III.3.2, Corollary to Proposition 2), or if the commutant is of the form $C \otimes F$, where C is an Abelian von Neumann algebra and F is a factor (Dixmier, 1959, III.3.2, Proposition 2) or if $\mathcal{M}$ is of type III and operates on a separable Hilbert space (Dixmier, 1959, III.8.6, Corollary 8).

Even if all α_g are spatial it is in general not possible to replace α by a unitary ray representation. Assume that each automorphism α_g can be implemented by a unitary operator U_g. By gluing together the U_g one obtains a unitary representation $U(G)$ *with multiplier in* $\mathcal{M}'$ (Streater, 1990). (Only if $\mathcal{M}$ is irreducible is this necessarily a unitary ray representation. If $\mathcal{M}$ is of type I, then the multipliers can be chosen to be in the center of $\mathcal{M}$.)

3. AN EXTENSION AND AN IMPRIMITIVITY THEOREM

Historical Remarks. Mackey's (1949, 1952) imprimitivity theorem says that a system of imprimitivity on a homogeneous space G/H is unitarily equivalent to a system of imprimitivity induced from a representation of H. Furthermore, there is a one-to-one correspondence of the equivalence classes of systems of imprimitivity based on G/H and the equivalence classes of unitary representations of the little group H. Blattner (1961, 1962) generalized the imprimitivity theorem to locally compact groups which are not necessarily separable. Takesaki (1968, 1973), Rieffel (1974), and Green (1980) gave generalizations of Mackey's theorem to C^*- and W^*-systems. They constructed W^*-systems which are induced from W^*-systems of a subgroup H and gave sufficient conditions for W^*-systems to be isomorphic to induced W^*-systems.

Naimark's (1943) extension theorem says that for any POV-measure a on a Borel space X in a Hilbert space $\mathcal{H}$ there is a unique minimal Hilbert space $\mathcal{H}'$ containing $\mathcal{H} = P\mathcal{H}'$ as a subspace and a PV-measure E on X in $\mathcal{H}'$ such that $a(\Delta) = PE(\Delta)P$. Stinespring (1955) proved an extension theorem for completely positive maps between C^*-algebras. Amann (1986, Theorem IV.2) extended Naimark's theorem to POV-measures on groups with values in von Neumann algebras and showed that covariance of the POV-measure implies covariance of the resulting PV-measure. In this context covariance properties are formulated not with respect to a unitary ray representation, but with respect to a representation of the group as automorphism group of the von Neumann algebra. The extension theorem to follow will generalize Amann's construction from POV-measures on groups to POV-measures on homogeneous spaces.

In Neumann (1972), Scutaru (1977), Cattaneo (1979), Castrigiano and Henrichs (1980), and Ali (1984) the results of Mackey and Naimark were combined to arrive at various versions of the following imprimitivity and extension theorems. Let a be a POV-measure on some G-space X in a Hilbert space $\mathcal{H}$ which transforms covariantly with respect to a unitary representation U of G. Then there is a bigger Hilbert subspace $\mathcal{H}'$ containing $\mathcal{H} = P\mathcal{H}'$

and a system of imprimitivity $(E, \hat{U})$ in $\mathcal{H}'$ such that U is a subrepresentation of $\hat{U}$ and $a(\Delta) = PE(\Delta)P$ for all Borel sets Δ of X. Furthermore, for any unitary ray representation U of G the following conditions are equivalent: (a) There exists a POV-measure a on G/H such that (a, U) is a system of covariance based on G/H. (b) U is unitarily equivalent to a subrepresentation of a representation of G induced from some unitary representation of H.

The Theorems. Now we will give an extension and an imprimitivity theorem for generalized systems of covariance. Theorem 1 generalizes Amann's (1986, Theorem IV.2) extension theorem, since it applies to covariant POV-measures on arbitrarily Borel G-spaces X, and not just to those on G. This generalization, however, is straightforward, because Amann's proof can be easily adapted.

Theorem 1. Let $(\mathcal{M}, G, \alpha)$ be a W^*-system, and (a, α) a generalized system of covariance based on X. Then there are:

- A W^*-system $(\mathcal{N}, G, \beta)$.
- A PV-measure E on X with values in $\mathcal{N}$ fulfilling the covariance condition

$$\beta_g(E(\Delta)) = E(g\Delta), \qquad \Delta \in \Sigma(X), \quad g \in G$$

- A projector P in the fixed-point algebra $\mathcal{N}^\beta := \{x \in \mathcal{N}: \beta_g(x) = x$ for all $g \in G\}$.
- An isomorphism $i{:}\mathcal{M} \to P\mathcal{N}P$ of the W^*-algebras $\mathcal{M}$ and $P\mathcal{N}P$ fulfilling

$$i(a(\Delta)) = PE(\Delta)P, \qquad \Delta \in \Sigma(X)$$

The projector P is an atom of $\mathcal{N}^\beta$ if α acts ergodically on $\mathcal{M}$.

The proof of this theorem can be found in Breuer (1994b, Section 2.5.1, Theorem 3).

Finally we will present an imprimitivity theorem for transitive generalized systems of covariance. This generalizes imprimitivity theorems for traditional systems of covariance in Neumann (1972), Scutaru (1977), Cattaneo (1979), Castrigiano and Henrichs (1980), and Ali (1984), because it applies to automorphic group representations and not only to projective unitary ones.

In order to formulate the theorem we first have to construct induced W^*-systems. In doing this I will follow Takesaki (1973). Let $(\mathcal{N}_0, H, \gamma)$ be a W^*-system of a closed subgroup H of G, where $\mathcal{N}_0$ is assumed to be isomorphic to a weakly closed subalgebra of bounded operators on a separable Hilbert space $\mathcal{H}_0$. We consider the tensor product $L^\infty(G) \otimes \mathcal{N}_0$ of $\mathcal{N}_0$ and the Abelian von Neumann algebra $L^\infty(G)$. The elements of $L^\infty(G) \otimes \mathcal{N}_0$ are regarded as bounded $\mathcal{N}_0$-valued functions x on G with the following properties.

1. For each pair ξ, $\eta \in \mathcal{H}_0$, the function $g \mapsto \langle \xi, x(g)\eta \rangle$ is Haar-measurable.

2. dg-ess $\sup_{g \in G} \|x(g)\| < \infty$.

On $L^\infty(G) \otimes \mathcal{N}_0$ define actions γ' of H and β of G by

$$(\gamma'_h(x))(s) := \gamma_h(x(sh)), \qquad s \in G, \quad h \in H, \quad x \in L^\infty(G) \otimes \mathcal{N}_0$$
$$(\beta_g(x))(s) := x(g^{-1}s), \qquad g, s \in G, \qquad x \in L^\infty(G) \otimes \mathcal{N}_0$$

Let $\mathcal{N}$ denote the fixed-point algebra of $L^\infty(G) \otimes \mathcal{N}_0$ under $\{\gamma'_h: h \in H\}$. Since γ'_h and β_g commute for all $h \in H$, $g \in G$, $\mathcal{N}$ is invariant under β_g. The restriction of β_g to $\mathcal{N}$ is also denoted by β_g. We say that the W^*-system $(\mathcal{N}, G, \beta)$ is *induced from the W^*-system* $(\mathcal{N}_0, H, \gamma)$ and write
$(\mathcal{N}, G, \beta) = \mathrm{Ind}_H^G(\mathcal{N}_0, H, \gamma)$.

Theorem 2. Let G be a locally compact separable group, and let H be a closed subgroup of G. Let a be a POV-measure based on G/H, and denote by $\mathcal{M}$ the von Neumann algebra $\{a(\Delta): \Delta \in \Sigma(G/H)\}''$. Let α be a pointwise σ-weakly continuous representation of G as automorphism group of $\mathcal{M}$. Assume that a is covariant with respect to α,

$$\alpha_g(a(\Delta)) = a(g\Delta), \qquad g \in G, \quad \Delta \in \Sigma(G/H)$$

Then there are:

- A W^*-system $(\mathcal{N}, G, \beta)$ with an Abelian von Neumann algebra $\mathcal{N}$.
- A PV-measure E on G/H generating $\mathcal{N} = \{E(\Delta): \Delta \in \Sigma(G/H)\}''$ and fulfilling the covariance condition

$$\beta_g(E(\Delta)) = E(g\Delta), \qquad \Delta \in \Sigma(G/H), \quad g \in G$$

- A projector P in the fixed-point algebra $\mathcal{N}^\beta$.
- An isomorphism i: $\mathcal{M} \to P\mathcal{N}P$ of the W^*-algebras $\mathcal{M}$ and $P\mathcal{N}P$ such that

$$i(a(\Delta)) = PE(\Delta)P, \qquad \Delta \in \Sigma(G/H)$$

Furthermore, there is a W^*-system $(\mathcal{N}_0, H, \gamma)$ such that $(\mathcal{N}, G, \beta)$ is isomorphic to $\mathrm{Ind}_H^G(\mathcal{N}_0, H, \gamma)$.

For a proof of this theorem see Breuer (1994b, Section 2.5.2, Theorem 4).

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